Probability of a Two-Hop Connection in a Random Mobile Network

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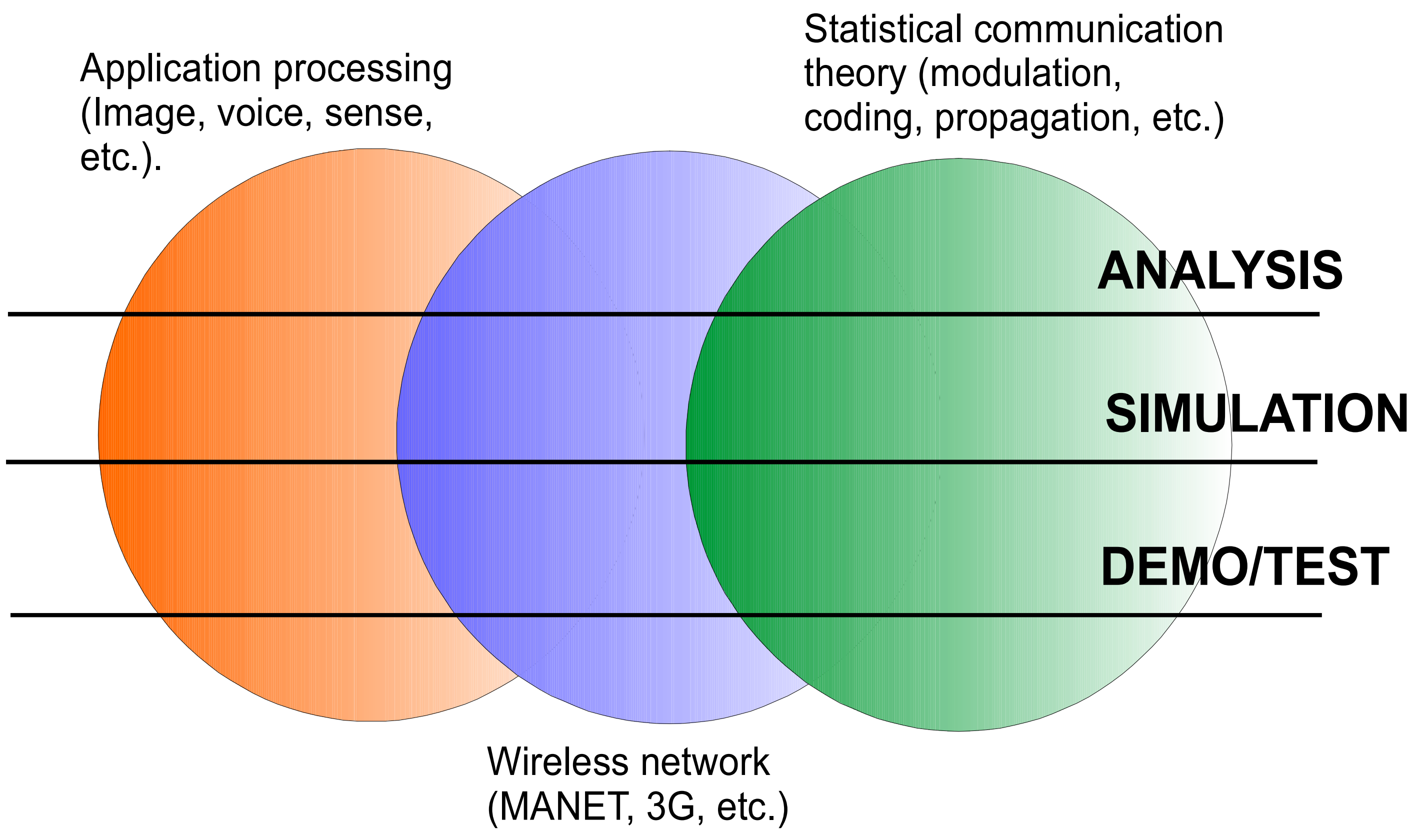
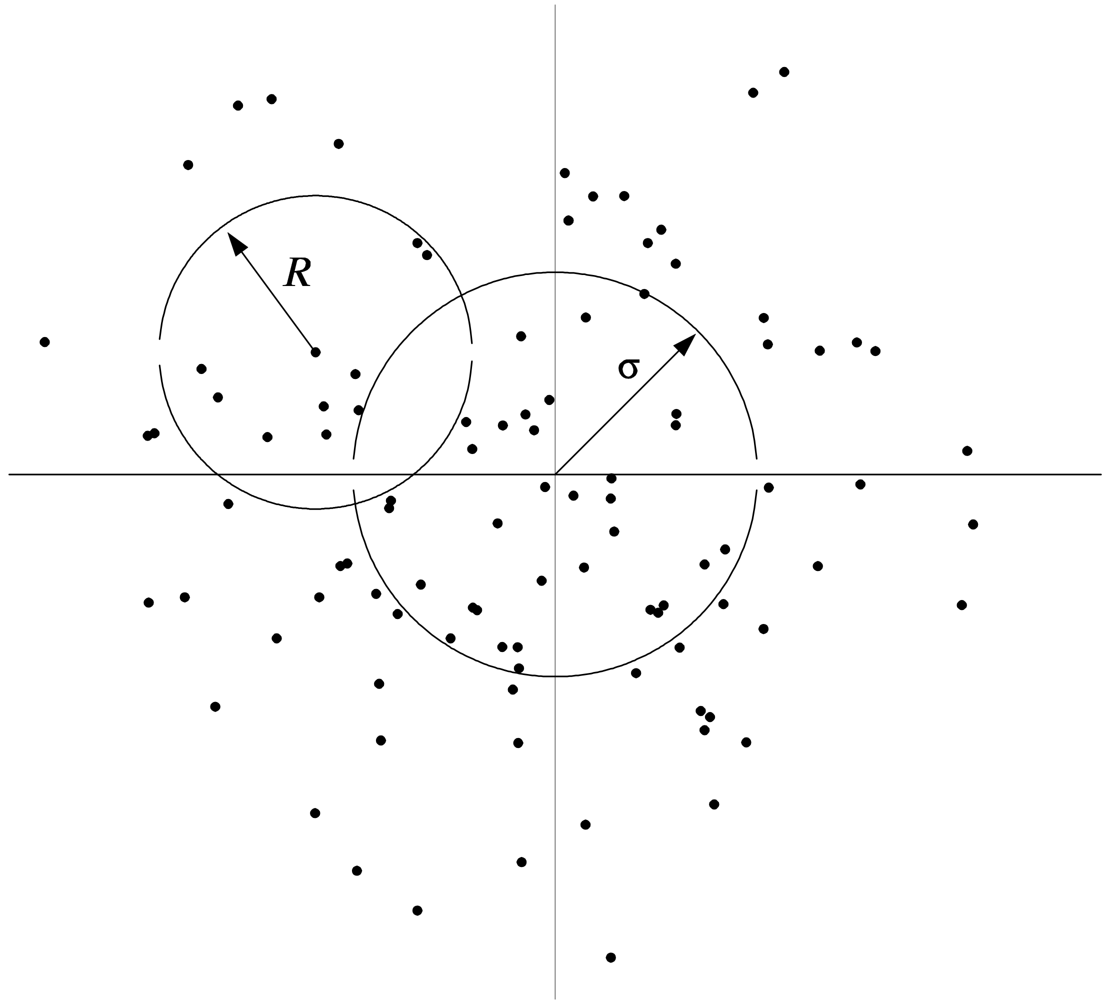
WCTG Work is Multidisciplinary



Random network with dispersion parameter σ, transmission range R



INTRODUCTION

- The probability that a link between two mobile radio terminals (nodes) has sufficient signal-to-noise ratio for acceptable transmission quality or reliability is:

$$\text{Pr}\{\text{Link is good}\} = \text{Pr}\{r \leq R\} = \text{F}_r(R)$$

- Assuming that different links are independent, the quantity $\text{F}_r(R)$ can be taken as the probability of success in a binomial trial in which two link endpoints are selected
  - If the trial is repeated K times, then an estimate of the number of good links is $K\,\text{F}_r(R)$.

- In general, an ad hoc mobile network is a multihop network in which the distance of a particular node A from some other nodes may be greater than R.

Hidden Terminal Problem

- Under a carrier sense multiple access procedure, the "hidden terminal problem" may arise, in which two nodes that are more than distance R apart transmit at the same time and their transmissions "collide" at a third node that is within distance R of both of the other nodes.

- Thus, all node pairs that are connected by a single relay node (are two hops away from each other) have the potential for creating a hidden terminal problem.

- In what follows, assuming a mobile network deployment in which the x and y coordinates of the mobile locations have Gaussian distributions, we calculate the probability that two nodes are connected by a two-hop path.

Node Distribution Assumptions and Single-Hop Connection Probability

- We assume that the x and y coordinates of any particular node are independent and have the following joint probability density function (pdf):

$$\begin{aligned}\mathsf{p}_{x,y}(x,\, y) &= \mathsf{p}_x(x)\, \mathsf{p}_y(y) \\ &= \frac{1}{2\pi\, \sigma_1\sigma_2} \exp\left\{ -\frac{1}{2}\left[\frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2}\right]\right\}, \qquad -\infty < x,\, y < \infty\end{aligned}$$

- The joint pdf of the x and y components $d_x = x_2 - x_1$ and $d_y = y_2 - y_1$ of the link distance between two nodes is given by

$$\mathsf{p}_{d_x,d_y}(d_x,\, d_y) = \frac{1}{4\pi\sigma_1\sigma_2} \exp\left\{ -\frac{1}{4}\left[\frac{d_x^2}{\sigma_1^2} + \frac{d_y^2}{\sigma_2^2}\right]\right\}$$

- The joint pdf of the link distance $r = \sqrt{d_x^2 + d_{y^2}}$ and orientation θ is found by using the transformation ($d_x = r\cos\theta$, $d_y = r\sin\theta$) to obtain

$$\mathsf{p}_{r,\theta}(r,\,\theta) = \frac{r}{4\pi\sigma_1\sigma_2}\,\mathsf{exp}\left\{-\frac{r^2}{4}\left[\frac{\cos^2\theta}{\sigma_1^2} + \frac{\sin^2\theta}{\sigma_2^2}\right]\right\}$$
$$= \frac{r}{4\pi\sigma_1\sigma_2}\,\mathsf{exp}\left\{-\frac{r^2}{8}\left[\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) + \left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right)\cos 2\theta\right]\right\}$$

- After integrating this pdf over the angle θ, the pdf for the link distance r is found to be

$$\mathsf{p}_r(r) = \frac{r}{2\sigma_1\sigma_2}\,\mathsf{exp}\left\{-\frac{r^2}{8}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)\right\}\,\mathsf{I}_0\left[\frac{r^2}{8}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right)\right]$$

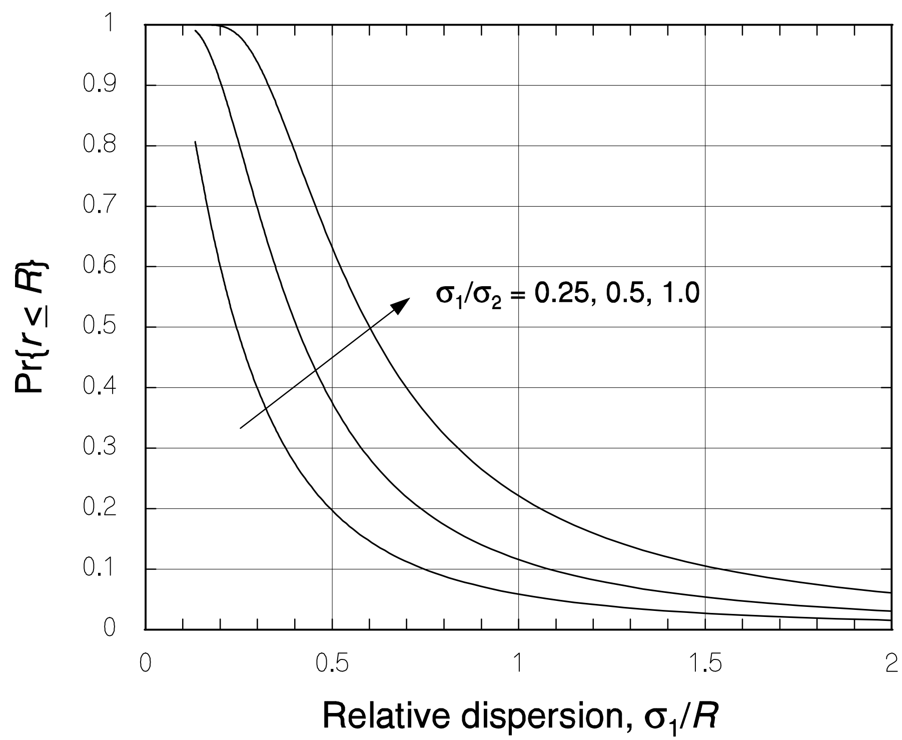
For $\sigma_1 = \sigma_2 = \sigma$, this simplifies to the expression

$$\mathsf{p}_r(r) = \frac{r}{2\sigma^2}\,\mathsf{e}^{-r^2/4\sigma^2}$$

Probability of a one-hop connection

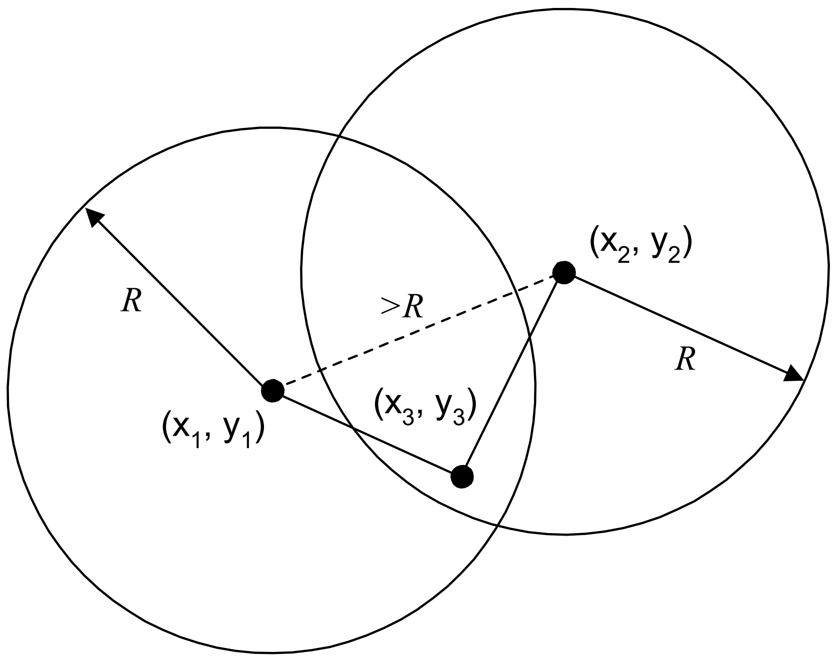
$$\text{Pr}\{1\text{-hop connection}\} = \text{Pr}\{r \le R\}$$

$$= \int_0^R dr\, \text{p}_r(r) = 1 - \text{e}^{-R^2/4\sigma^2} \quad \text{for } \sigma_1 = \sigma_2 = \sigma$$



ANALYSIS OF TWO-HOP CONNECTIVITY

- A two-hop connection between nodes 1 and 2 exists if two conditions are met:
  - (1) the positions (x_1, y_1) for node 1 and (x_2, y_2) for node 2 are such that the distance between the nodes is greater than the transmission range R but less than $2R$; and
  - (2) the position (x_3, y_3) for at least one other node is within the distance R of both nodes 1 and 2.

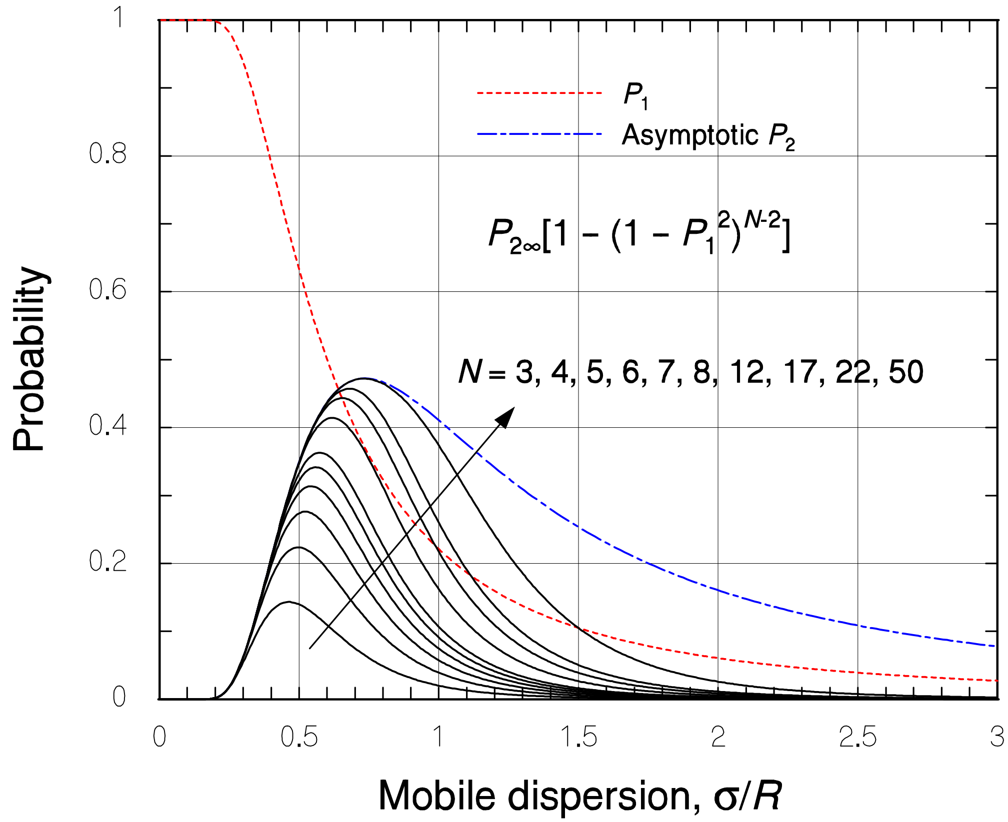


NAIVE APPROXIMATION

- A "naive" approximation to the two-hop connection probability:

$$P_2 = \text{Pr}\{R < d_{12} < 2R\} \times \left[1 - (1 - \text{Pr}\{d_{31} < R \cap d_{32} < R\})^{N-2}\right]$$

$$\approx P_{2\infty} \times \left[1 - (1 - P_1^2)^{N-2}\right]$$



- The probability of a two-hop connection for the case of $N = 3$ nodes:

$$\text{Pr}\{1 \to 2 \text{ in } 2 \text{ hops}\} = \text{Pr}\{R < r < 2R \text{ and node } 3 \text{ in the area of intersection}\}$$

$$= \underbrace{\int dx_1 \int dy_1 \int dx_2 \int dy_2}_{R<r<2R} \underbrace{\int dx_3 \int dy_3}_{A(x_1,\, y_1,\, x_2,\, y_2)} \; \mathsf{p}_{x,y}(x_1, y_1, x_2, y_2, x_3, y_3)$$

$$= \underbrace{\int dx_1 \int dy_1 \int dx_2 \int dy_2}_{R<r<2R} \mathsf{p}_{x,y}(x_1, y_1, x_2, y_2) \underbrace{\int dx_3 \int dy_3}_{A(x_1,\, y_1,\, x_2,\, y_2)} \; \mathsf{p}_{x,y}(x_3, y_3)$$

- Approximation:

$$\underbrace{\int dx_3 \int dy_3}_{A(x_1,\, y_1,\, x_2,\, y_2)} \; \mathsf{p}_{x,y}(x_3, y_3) \approx \mathsf{p}_{x,y}\left(\tfrac{x_1+x_2}{2}, \tfrac{y_1+y_2}{2}\right) \underbrace{\int dx_3 \int dy_3}_{A(x_1,\, y_1,\, x_2,\, y_2)}$$

where

$$\underbrace{\int dx_3 \int dy_3}_{A(x_1,\, y_1,\, x_2,\, y_2)} = R^2 \left(2\cos^{-1}\left(\frac{r}{2R}\right) - \sin\left[2\cos^{-1}\left(\frac{r}{2R}\right)\right]\right) \triangleq B(r)$$

- For more than $N = 3$ nodes:

$$\mathrm{Pr}\{1 \to 2 \text{ in } 2 \text{ hops}\} = \mathrm{Pr}\{R < r < 2R \text{ and } \geq 1 \text{ node in area of intersection}\}$$

$$= \mathrm{Pr}\{R < r < 2R \text{ and NOT } (\text{no node in area of intersection})\}$$

$$= \underbrace{\int dx_1 \int dy_1 \int dx_2 \int dy_2}_{R<r<2R} \; \mathsf{p}_{x,y}(x_1,\, y_1,\, x_2,\, y_2)$$

$$\times \left[1 - \left[1 - \underbrace{\int dx_3 \int dy_3}_{A(x_1,\, y_1,\, x_2,\, y_2)} \; \mathsf{p}_{x,y}(x_3,\, y_3)\right]^{N-2}\right]$$

$$\approx \underbrace{\int dx_1 \int dy_1 \int dx_2 \int dy_2}_{R<r<2R} \; \mathsf{p}_{x,y}(x_1,\, y_1,\, x_2,\, y_2)$$

$$\times \left[1 - \left[1 - \mathsf{p}_{x,y}\left(\tfrac{x_1+x_2}{2},\, \tfrac{y_1+y_2}{2}\right) B(r)\right]^{N-2}\right]$$

$$\text{Pr}\{1 \to 2 \text{ in } 2 \text{ hops}\} \approx \int_R^{2R} dr \int_0^\infty d\rho \, \frac{r\,\rho}{4\,\sigma^4} \, \text{exp}\left\{ -\frac{r^2+\rho^2}{4\sigma^2} \right\} \times \left[1 - \left[1 - \frac{B(r)}{2\pi\sigma^2} \, \text{e}^{-\rho^2/8\sigma^2} \right]^{N-2} \right]$$

$$= \int_R^{2R} dr \, \frac{r}{2\sigma^2} \, \text{e}^{-r^2/4\sigma^2} \sum_{n=1}^{N-2} \binom{N-2}{n} \left(\frac{B(r)}{2\pi\sigma^2} \right)^n \frac{(-1)^{n+1}}{1+n/2}$$

$$= \sum_{n=1}^{N-2} \binom{N-2}{n} \frac{(-1)^{n+1}}{1+n/2} \, \frac{1}{(2\pi)^n \gamma^{2n}} \times \int_{1/4\gamma^2}^{1/\gamma^2} d\nu \, \text{e}^{-\nu} \left[2\,\text{cos}^{-1} \gamma\sqrt{\nu} - \text{sin}\left(2\,\text{cos}^{-1} \gamma\sqrt{\nu} \right) \right]^n$$

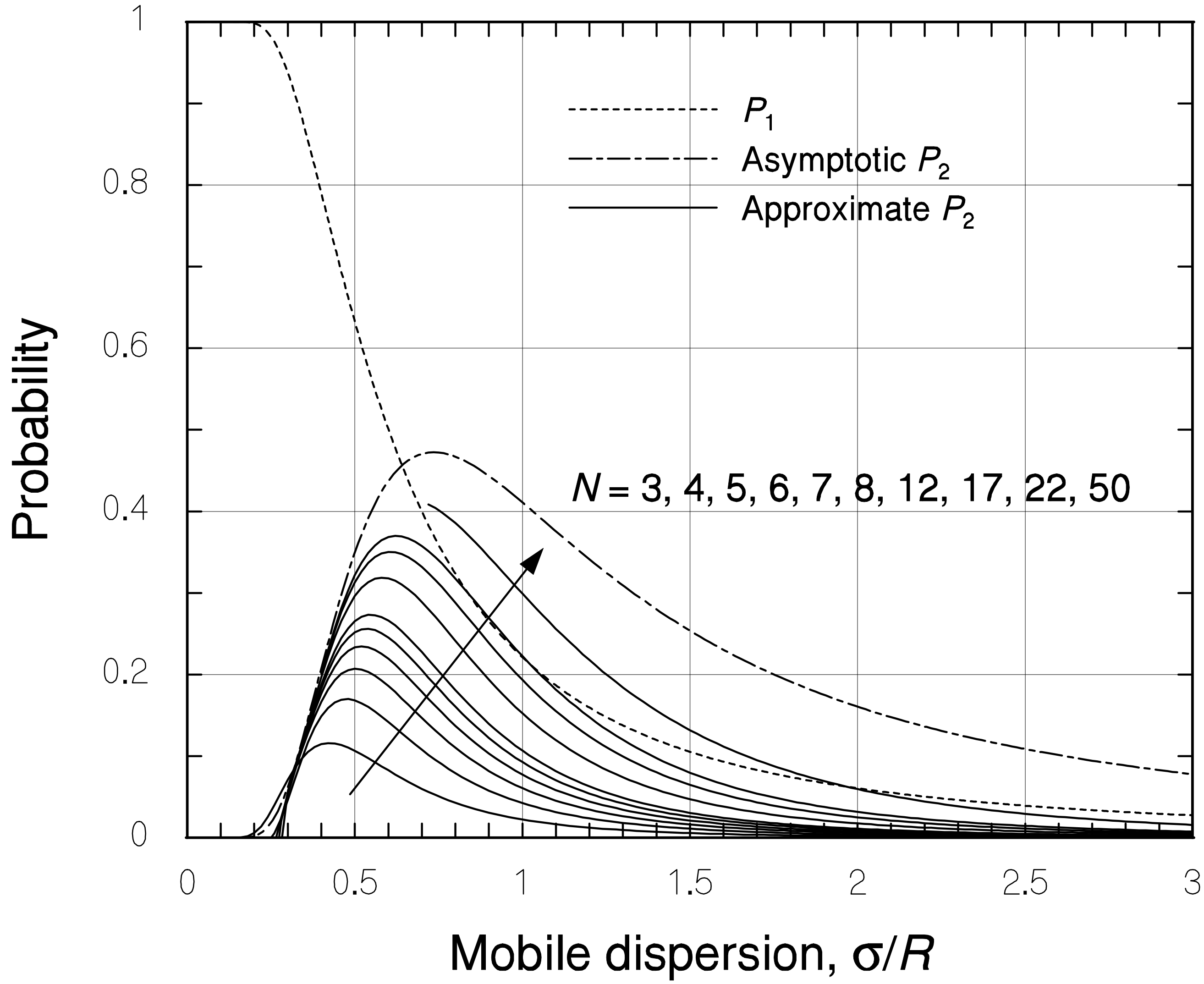
Upper Bound on the Two-Hop Connection Probability

- An upper bound on the two-hop connection probability can be derived by considering the effect as the number of nodes approaches infinity.

$$\mathsf{Pr}\{1 \to 2 \text{ in } 2 \text{ hops}\} < \underbrace{\int dx_1 \int dy_1 \int dx_2 \int dy_2}_{R < r < 2R} \; \mathsf{p}_{x,y}(x_1,\, y_1,\, x_2,\, y_2) \times [1-0]$$

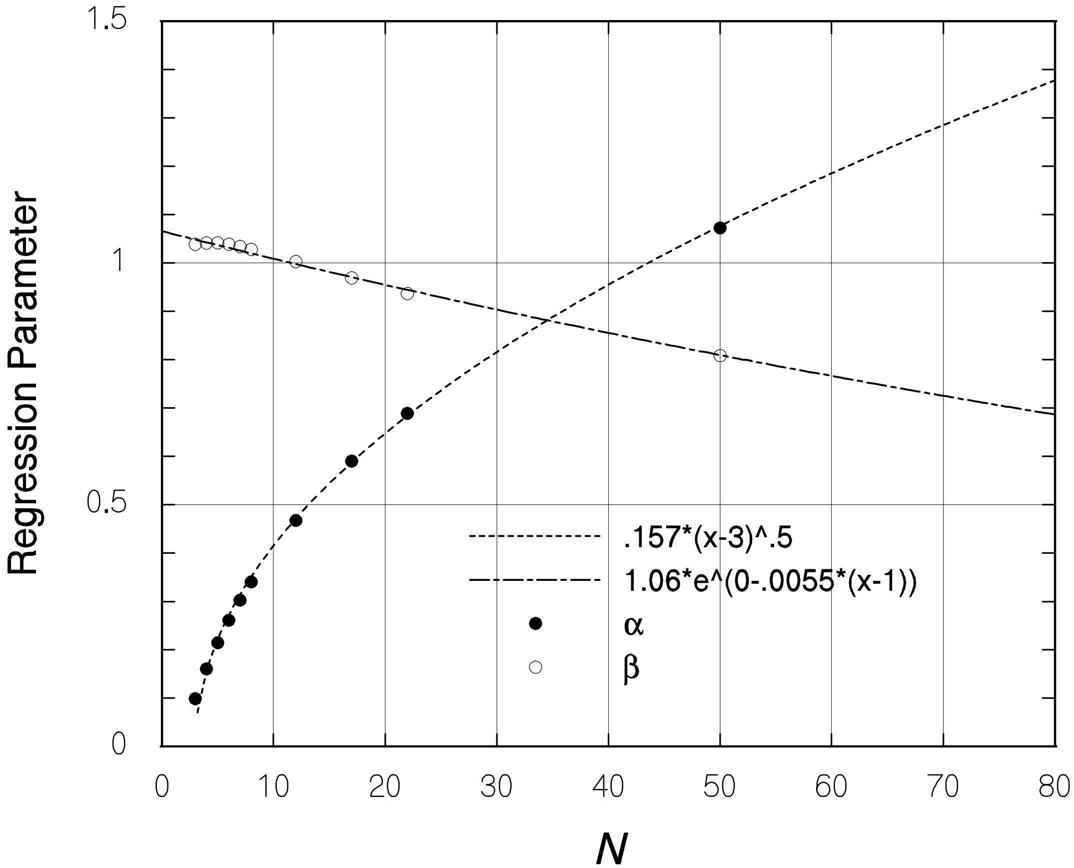
$$= \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_R^{2R} dr \int_0^{\infty} d\rho \; \frac{\frac{1}{4}\, r\, \rho}{(2\pi\sigma^2)^2} \,\mathsf{exp}\left\{-\frac{r^2+\rho^2}{4\sigma^2}\right\}$$

$$= \int_R^{2R} dr\, \frac{r}{2\sigma^2}\, \mathsf{e}^{-r^2/4\sigma^2} = \mathsf{e}^{-R^2/4\sigma^2} - \mathsf{e}^{-R^2/\sigma^2}$$


P1
Asymptotic P2
Approximate P2
N = 3, 4, 5, 6, 7, 8, 12, 17, 22, 50
Probability
Mobile dispersion, σ/R
0
0.2
0.4
0.6
0.8
1
0.5
1.5
2
2.5
3

- A nonlinear regression was used to fit the two-hop connection probability results to curves of the form given by the following equation, parametric in α and β:

$$f(\gamma;\,\alpha,\,\beta) = \left(1 - \alpha\,\mathrm{e}^{-\beta/\gamma^2}\right) P_{2\infty}, \quad \text{using } \frac{\sigma}{R} \triangleq \gamma$$

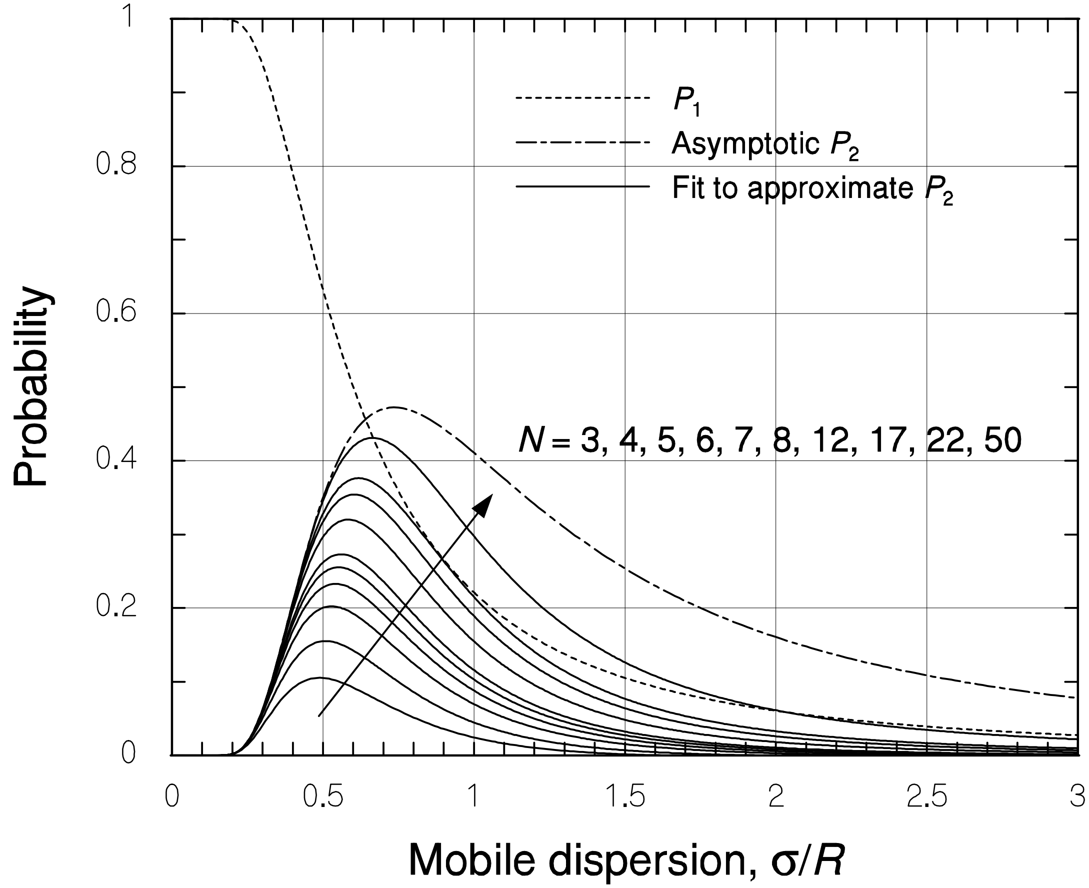


- Regressions and curve fitting → empirical expression for P_2:

$$P_2 \approx \left(1 - 1.0383\,\mathrm{e}^{-0.0988/\gamma^2}\right) P_{2\infty}, \quad N = 3$$

and

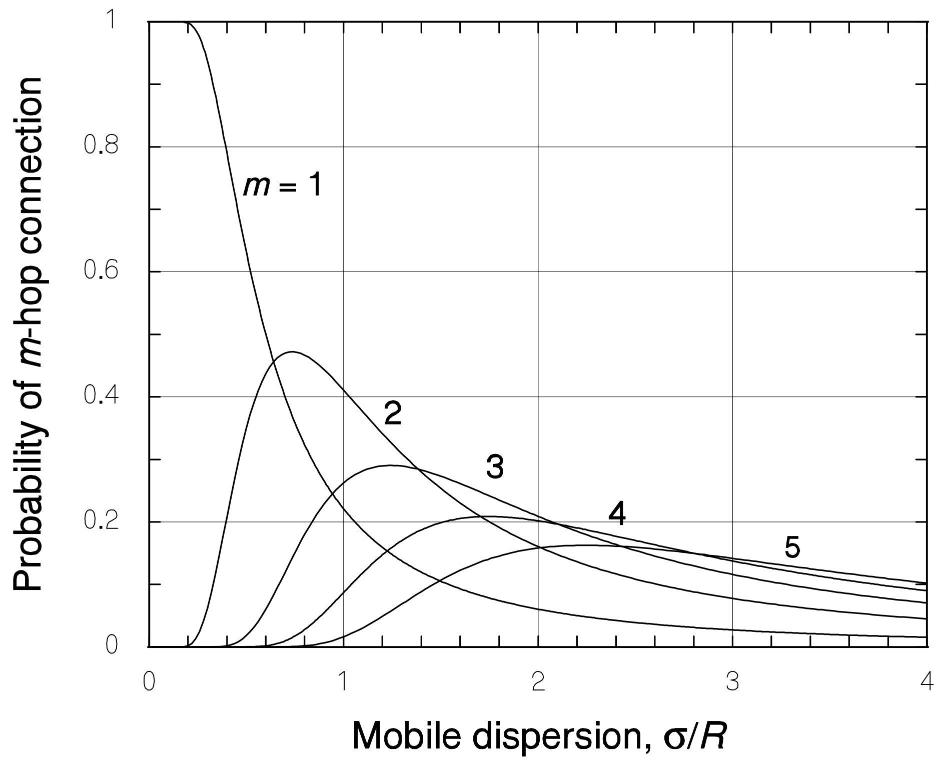
$$P_2 \approx \left(1 - 1.06\,\mathrm{e}^{-0.0055(N-1) - 0.157\sqrt{N-3}\,/\gamma^2}\right) P_{2\infty},\ N > 3$$



Asymptotic m-Hop Connection Probability and Applications

- The form of the asymptotic probability of a two-hop connection suggests the following form for the asymptotic probability of an m-hop connection:

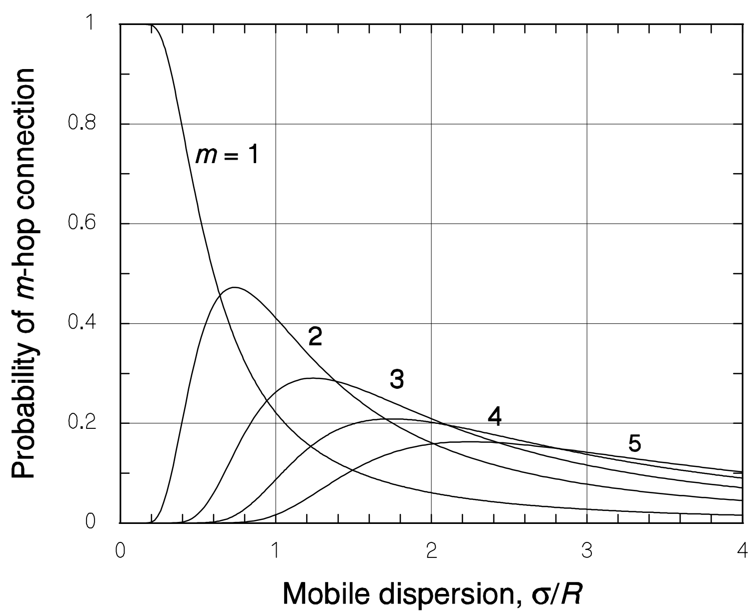
$$P_m < \Pr\{(m-1)R < r < mR\} = \mathrm{e}^{-(m-1)^2R^2/4\sigma^2} - \mathrm{e}^{-m^2R^2/4\sigma^2}$$



Most Likely Number of Hops

- The most likely number of hops, given the value of γ, the relative dispersion of the nodes:

$$\text{Most likely \# hops} = \begin{cases} 1, & \sigma/R < 0.64 \\ 2, & 0.64 < \sigma/R < 1.39 \\ 3, & 1.39 < \sigma/R < 2.10 \\ 4, & 2.10 < \sigma/R < 2.81 \\ 5, & 2.81 < \sigma/R < 3.52 \end{cases}$$



Upper Bound on Average Number of Hops

- An upper bound on the average number of hops between node pairs is given by the expectation

$$\mathsf{E}\{h\} = \sum_{m=1}^{N-1} m\, P_m < \sum_{m=1}^{\infty} m\, P_{m\infty}$$

or

$$\begin{aligned}\mathsf{E}\{h\} &< \sum_{m=1}^{\infty} m \left(\mathsf{e}^{-(m-1)^2/4\gamma^2} - \mathsf{e}^{-m^2/4\gamma^2} \right) \\ &= \sum_{m=0}^{\infty} (m+1)\mathsf{e}^{-m^2/4\gamma^2} - \sum_{m=0}^{\infty} m\, \mathsf{e}^{-m^2/4\gamma^2} \\ &= \sum_{m=0}^{\infty} \mathsf{e}^{-m^2/4\gamma^2} \triangleq \overline{h}_+(\gamma)\end{aligned}$$

- Except for small γ ($\sigma < R/4$), this function is practically a linear function of γ!

Upper bound on average number of hops (continued)

- Derivation of the linear form for $\mathsf{E}\{h\}$:

$$\sum_{m=0}^{\infty} \mathsf{e}^{-m^2/4\gamma^2} = \tfrac{1}{2} + \tfrac{1}{2}\int_{-\infty}^{\infty} dx\, \mathsf{e}^{-x^2/4\gamma^2} g(x)$$

where $g(x)$ is a periodic train of impulses, with the Fourier series representation (using $i = \sqrt{-1}$)

$$g(x) = \sum_{m=-\infty}^{\infty} \delta(x-m) = \sum_{k=-\infty}^{\infty} \mathsf{e}^{2\pi k i x}$$

- Substituting in the integral, then exchanging the order of summation and integration, yields an expression involving the characteristic function of a Gaussian random variable that reduces to

$$\begin{aligned}\overline{h}_{+}(\gamma) &= \tfrac{1}{2} + \gamma\sqrt{\pi}\sum_{k=-\infty}^{\infty} \mathsf{e}^{-(2\pi k\gamma)^2} \\ &\approx 0.5 + 1.772\,\gamma, \quad \gamma > 0.5\end{aligned}$$